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OPTIMAL CONTROL OF PARTIALLY OBSERVABLE STOCHASTIC SYSTEMS WITH AN EXPONENTIAL-OF-INTEGRAL PERFORMANCE INDEX

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Optimal control of partially observable stochastic systems with an exponential-of-integral performance index*)

by

A. Bensoussan**) & J.H. van Schuppen

ABSTRACT

The stochastic control problem with linear stochastic differential equations driven by Brownian motion processes and as cost functional the exponential of a quadratic form, is considered. The solution is shown to exist of a linear control law, and of a linear stochastic differential equation which has the same structure as the Kalman filter but depends explicitly on the cost functional. The separation property does not hold in general for the solution to this problem.

KEY WORDS & PHRASES: *Stochastic control, linear systems, exponential-of-integral cost functional, Linear-Exponential-Gaussian problem*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

The class of so called Linear-Exponential-Gaussian (LEG) stochastic control problems has been introduced by Jacobson [3] and Speyer et al. [6]. Since then several papers have presented solutions to special cases of this problem. Below the solution to the general case of a partially observed stochastic system is presented.

The simplest special case of the LEG stochastic control problem is that of the completely observable system

$$(1.1)\qquad dx = (Fx+Bv)dt + Gdw, \quad x_0 = \mu_0,$$

with the cost functional

$$(1.2)\qquad J(v(\cdot)) = E[\mu \exp((\mu/2)[Mx_{t_1}^2 + \int_0^{t_1} (Qx^2+Nv^2)dt])].$$

Under certain definite conditions there exists a linear optimal control which is implementable by a finite dimensional system, see [3].

Subsequently Speyer et al. [6, 7] have considered the case of a partially observed system

$$(1.3)\qquad \begin{aligned} dx &= (Fx+Bv)dt + Gdw, \quad x_0 = \mu_0, \\ dy &= H x\,dt + R^{\frac{1}{2}}db, \quad y_0 = 0, \end{aligned}$$

with the cost functional (1.2) with $Q = 0$. Again there exists a finite dimensional implementable optimal control.

Yet another case of a partially observed stochastic control problem is considered by Kumar, van Schuppen [5]. There the general cost functional (1.2) is combined with the stochastic system (1.3), but with $G = 0$. It is proven that the optimal control is given by

$$(1.4)\qquad u_t = -N^{-1}(t)B^*(t)[L(t)\hat{x}_t + M(t)\eta(t)]$$

where $\hat{x}_t$ is produced by the Kalman filter and

$$\eta(t) = \int_0^t K(t,s)u_s ds.$$

The control is thus implementable by a finite dimensional system. It has long been thought that the general case does not have a finite dimensional implementable optimal controller.

In this paper the solution to the general stochastic control problem will be presented, consisting of the stochastic system (1.3) and the cost functional (1.2). It will be proven that the optimal control is given by

$$(1.5) \qquad u_t = -N^{-1}(t)B^*(t)S(t)r_t$$

where

$$(1.6) \qquad dr = [F - PH^*R^{-1}H + \mu PQ]r_t dt + Budt + PH^*R^{-1}dy, \quad r_0 = \mu_0,$$

and P is the solution of

$$(1.7) \qquad \dot{P} - FP - PF^* + P(H^*R^{-1}H-\mu Q)P - GG^* = 0, \quad P(0) = P_0.$$

The recursions for the sufficient statistics reduce to the Kalman filter if $Q = 0$. The representation of the solution (1.5) and (1.6) is much more convenient then that given by (1.4).

The motivation for considering LEG stochastic control problems is that for certain applications the exponential-of-integral cost functional may be better suitable than the usual quadratic cost functional. The reason for this is that the exponential form introduces a nonlinear relation between small and large deviations from the equilibrium state. The economic interpretation of the solution of the LEG problem is discussed by van der Ploeg [9]. One may interpret the solution as an attitude of either risk-preference, for $\mu < 0$, or of risk-aversion, for $\mu > 0$, see [9, 10].

There is also a system theoretic motivation to consider LEG stochastic control problems. A major question in stochastic control theory is to classify those stochastic control systems and cost functionals that lead to finite dimensional control algorithms. An attempt to define a finite dimensional control algorithm will not be given here, but the solution presented by (1.5) and (1.6) illustrates what the authors have in mind. It is expected that the availability of the solution to the LEG problem may provide insight into the above stated question. Apparently the invariance of the conditional cost functional plays a key role. In addition the solution provides an example of a sufficient statistic for a stochastic control problem which does not have the separation property.

A brief summary of the paper follows. In the following section a problem formulation is given. In section 3 an equivalent expression for the cost functional is derived. The solution is presented in section 4.

2. THE PROBLEM FORMULATION

Notations

Let (Ω, A, P) be a complete probability space and $T = [0, t_1]$, on which are defined

(2.1) $w: \Omega \times T \to R^k$ a standard Wiener process;

$\tilde{b}: \Omega \times T \to R^d$ a standard Wiener process;

$R: T \to R^{d \times d}$ a symmetric positive definite matrix for which there exists a $r_0 \in (0, \infty)$ such that for all $t \in T$ $R(t) \geq r_0 I$;

$y: \Omega \times T \to R^d$

(2.2) $$dy_t = R^{\frac{1}{2}}(t) d\tilde{b}_t, \quad y_0 = 0;$$

(2.3) $x_0: \Omega \to R^n$, a Gaussian random variable with mean μ_0 and variance P_0, with P_0 non-singular.

Assume that x_0, y, w are independent objects. The process y will be termed the *observation process*.

Consider processes $v: \Omega \times T \to R^m$ which belong to $L_y(0, t_1; R^m)$, meaning that they are adapted to the σ-algebra family $Y^t = \sigma(\{y_s, \forall s \leq t\})$ generated by the observation process. For uninitiated readers of stochastic control the well known fact is pointed out that v in $L_y(0, t_1; R^m)$ is equivalent to specifying a non-anticipating function $f: (R^d)^T \to R^m$ such that $v_t = f(y(\cdot))$. Here non-anticipating means that for any $t \in T$ $v_t = f(y(\cdot))$ depends only on the path of y before time t. One may consider f to be a control law.

For $v \in L_y(0, t_1; R^m)$ define the *state process* as the solution of the stochastic differential equation

(2.4) $$dx_t = [F(t)x_t + B(t)v_t]dt + G(t)dw_t, \quad x_0,$$

where

$$F: T \to R^{n\times n}, \quad B: T \to R^{n\times m}, \quad G: T \to R^{n\times k}.$$

Define the process $b: \Omega \times T \to R^d$

(2.5) $$b_t = \tilde{b}_t - \int_0^t R^{-\frac{1}{2}}(s)H(s)x_s \quad ds$$

for $H: T \to R^{d\times n}$. Define the change of probability

(2.6) $$d\tilde{P}/dP = \exp\left(\int_0^{t_1} R^{-\frac{1}{2}}Hx.d\tilde{b} - \tfrac{1}{2}\int_0^{t_1} H^*R^{-1}Hx.xds\right)$$
$$= \exp\left(\int_0^{t_1} R^{-1}Hx.dy - \tfrac{1}{2}\int_0^{t_1} H^*R^{-1}Hx.xds\right).$$

Since the integrand entering into the stochastic integral at the right hand side of (2.6) is unbounded, an assumption is necessary to ensure that $\tilde{P}$ is indeed a probability measure. The following criterion will be used, see [I.I. Gikhman, A.V. Skorokhod, 2, p. 83]: there exist μ, $c \in (0,\infty)$ such that

(2.7) $$E[\exp(\mu H^*R^{-1}Hx_t.x_t)] \le c$$

for all $t \in T$. Define $x_1: \Omega \times T \to R^n$, $x_2: \Omega \times T \to R^n$

(2.8) $$\dot{x}_{1t} = Fx_{1t} + Bv_t, \quad x_{10} = 0;$$

(2.9) $$dx_{2t} = Fx_{2t}\, dt + Gdw_t, \quad x_{20} = x_0.$$

Then

(2.10) $$x_t = x_{1t} + x_{2t},$$

and x_1, x_2 are independent. Therefore (2.7) can be majorized as follows

$$E[\exp(\mu H^*R^{-1}H(x_1+x_2)^2)]$$
$$\le E[\exp(2\mu H^*R^{-1}H(x_1^2+x_2^2)]$$
$$\le E[\exp(2\mu H^*R^{-1}Hx_{1t}^2)]E[\exp(2\mu H^*R^{-1}Hx_{2t}^2)].$$

Since x_{2t} is a Gaussian random variable, it is possible to find a $\mu \in (0,\infty)$ such that the second expectation is finite. Because

$$\|x_{1t}\|^2 \le c \int_0^{t_1} \|v_s\|^2 ds,$$

(2.7) will be satisfied if there exists a $\nu \in (0,\infty)$ such that

$$E[\exp(\nu \int_0^{t_1} \|v_s\|^2 ds)] < \infty. \qquad (2.11)$$

The parameter ν may depend on the control, but not on Ω. The preliminary set of admissable controls is therefore defined as

$$\underline{U}_1 = \{v \in L_y(0,t_1;R^m) \mid \exists \nu \in (0,\infty) \text{ such that (2.11) holds}\}.$$

Other restrictions will be stated later.

With respect to the probability measure $\tilde{P}$ the process b is a standard Wiener process, x_0,b,w are independent objects, and

$$dy_t = H(t)x_t dt + R^{\frac{1}{2}}(t)db_t, \quad y_0 = 0. \qquad (2.12)$$

Furthermore the measures P and $\tilde{P}$ are identical with respect to x_0,w.

In the rest of the paper the time parameter will often be suppressed.

<u>The stochastic control problem</u>

Because of the dependence of $\tilde{P}$ on the control v the notation $\tilde{P}^v$ will be used. Consider the cost functional

$$J(v(\cdot)) = \tilde{E}^v[\mu\exp((\mu/2)[Mx_{t1}^2 + \int_0^{t_1} (Qx^2+Nv^2)ds])] \qquad (2.13)$$

where $\mu \in R$, $\mu \neq 0$, is given,

$$(2.14) \quad \begin{cases} M \in R^{n\times n} \text{ is symmetric and non-negative definite,} \\ Q: T \to R^{n\times n} \text{ is also symmetric and non-negative definite, and} \\ N: T \to R^{m\times m} \text{ is symmetric positive definite for which there} \\ \quad \text{exists a } n_0 \in (0,\infty) \text{ such that for all } t \in T\ N(t) \ge n_0 I. \end{cases}$$

Note that for both $\mu > 0$ and $\mu < 0$ $J(v(\cdot))$ should be minimized.

In order that $J(v(\cdot))$ is finite for at least some $v \in \underline{U}_1$ an assumption is necessary. Let

$$(2.15) \qquad \underline{U}_2 = \{u \in \underline{U}_1 \mid J(u(\cdot)) < \infty\}$$

to be called the *clas of admissable controls*. It will be assumed that $\underline{U}_2$ is non-empty. A condition guaranteeing that $\underline{U}_2$ is non-empty is the following. Let $v \equiv 0$. Then $x = x_2$, and in this case the probability laws of x_2 with respect to P and $\tilde{P}^v$ are the same. Therefore

$$(2.16) \qquad J(0(\cdot)) = E[\mu \exp((\mu/2)[Mx_{t1}^2 + \int_0^{t_1} Qx_{2\,s}^2 ds])].$$

If $J(0(\cdot)) < \infty$, then $\underline{U}_2$ is non-empty, and it is likely that it will contain more then one element. One can also reformulate (2.13) as

$$(2.17) \qquad J(v(\cdot)) = E[\mu \exp((\mu/2)[Mx_{t1}^2 + \int_0^{t_1} Nv^2 ds]$$
$$+ \int_0^{t_1} (\mu Q/2 - H^* R^{-1} H/2) x^2 ds + \int_0^{t_1} R^{-1} Hx_s . dy_s].$$

DEFINITION 2.1. a. An admissable control u^* will be called *optimal* for the cost functional (2.13) if

$$(2.18) \qquad J(u^*(\cdot)) \le J(v(\cdot))$$

for all $v \in \underline{U}_2$. Here the state process and the observation process are given respectively by (2.4) and (2.12).

b. An admissable control u^* will be called *conditional optimal* for the cost functional (2.13) if for all $t \in T$

$$(2.19) \qquad \tilde{E}^{u*}[c^{u*} \mid Y^t] \le \tilde{E}^v[c^v \mid Y^t]$$

for all $v \in \underline{U}_2$ such that for all $s \le t$ $u_s^* = v_s$. Here

$$(2.20) \qquad c^v = \mu\exp((\mu/2)[Mx_{t1}^2 + \int_0^{t_1} (Qx^2+Nv^2)ds]).$$

The definition of conditional optimality is due to C. Striebel [8, Ch. 4]. If $u^*\epsilon \underline{U}_2$ is conditionally optimal then it is also optimal; take t = 0 in (2.19) and use $y_0 = 0$. However the converse is not true, see [5, p. 315] for a counterexample.

PROBLEM 2.2. The *Linear-Exponential-Gaussian stochastic control problem* is to determine an admissable control u_1^* that is optimal, and an admissable control u_2^*, possibly different from u_1^*, that is conditionnally optimal. The state process and the observation process are given respectively by (2.4) and (2.12).

3. CALCULATION OF THE COST FUNCTION

The solution to the stochastic control problem 2.2. that will be given in section 4 is based on an alternate expression for the cost functional. This result will be derived below.

Definitions.

The following variables are introduced:

$P: T \to R^{n\times n}$

$$(3.1) \qquad \dot{P} - FP - PF^* + P(H^*R^{-1}H-\mu Q)P - GG^* = 0, \quad P(0) = P_0;$$

$r: \Omega \times T \to R^n$

$$(3.2) \qquad dr = [F - PH^*R^{-1}H + \mu PQ]rdt + Bvdt + PH^*R^{-1}dy, \quad r_0 = \mu_0;$$

for any $v \in \underline{U}_2$ $\pi^v: \Omega \times R^n \times T \to R$

$$(3.3) \qquad \pi^v(x,t) = \exp(-\tfrac{1}{2}P(t)^{-1}(x-r)\cdot(x-r) + \int_0^t R^{-1}Hr.dy - \tfrac{1}{2}\int_0^t H^*R^{-1}Hr.rds + (\mu/2)\int_0^t (Qr^2+Nv^2)ds + (\mu/2)\int_0^t tr(PQ)ds)(2\pi)^{-n/2}|P(t)|^{-\frac{1}{2}};$$

(3.4) $$K(v(\cdot)) = E[\mu \int \exp((\mu/2)Mx^2)\pi^v(x,t_1)dx];$$

$\Sigma: T \to R^{n\times n}$

(3.5) $$\dot{\Sigma} - \Sigma GG^*\Sigma + F\Sigma + \Sigma F^* - \mu Q + H^*R^{-1}H = 0, \quad \Sigma(t_1) = -\mu M.$$

ASSUMPTIONS 3.1.

(3.6) $H^*R^{-1}H - \mu Q \geq 0$ (then a solution to (3.1) exists);

(3.7) $P(t) \geq c_1 I$, for some $c_1 \in (0,\infty)$ and for all $t \in T$;

(3.8) $P^{-1}(t) + \Sigma(t) > 0$ for all $t \in T$

THEOREM 3.2. *Assume that* (3.6), (3.7), *and* (3.8) *hold. Assume further that the Riccati equation* (3.5) *has a symmetric bounded solution. For any control* v *in the class of admissable controls* $\underline{U}_2$ *one has the equality*

(3.9) $$J(v(\cdot)) = K(v(\cdot))$$

where $J(v(\cdot))$ *is defined by* (2.13) *and* $K(v(\cdot))$ *by* (3.4).

The proof of theorem 3.2 is based on several lemma's.

Preliminary calculations

It will be convenient to introduce the processes

(3.10) $$z_t = \exp(\int_0^t R^{-1}Hx.dy - \tfrac{1}{2}\int_0^t H^*R^{-1}Hx.xds),$$
$$\lambda_t = \exp((\mu/2)\int_0^t (Qx^2 + Nv^2)ds).$$

Thus

(3.11) $$J(v(\cdot)) = E[\mu \exp((\mu/2)Mx_{t1}^2)\lambda_{t1} z_{t1}].$$

Define

$$(3.12)\qquad s_t = P^{-1}(t)r_t^2 - 2\int_0^t R^{-1}Hr.dy + \int_0^t H^*R^{-1}Hr.rds$$

$$-\mu\int_0^t (Qr^2+Nv^2)ds - \mu\int_0^t tr(PQ)ds + \ln((2\pi)^n|P(t)|).$$

Then

$$(3.13)\qquad \pi^v(x,t) = \exp(-\tfrac{1}{2}[P^{-1}(t)x.x - 2P^{-1}(t)r_t.x + s_t]).$$

The following result is then obtained.

LEMMA 3.3. *The process* s *is a solution of the differential equation*

$$(3.14)\qquad ds_t/dt = tr(G^*P^{-1}G+2F) - |G^*P^{-1}r|^2 + 2P^{-1}Bv.r - \mu Nv^2.$$

PROOF. One uses

$$dP^{-1}r^2 = (P^{-1})'r.rdt + 2P^{-1}r.dr + tr(PH^*R^{-1}H)dt,$$

$$d(P^{-1})' = - P^{-1}\dot{P}P^{-1}$$

$$= - P^{-1}F - F^*P^{-1} + H^*R^{-1}H - \mu Q - P^{-1}GG^*P^{-1},$$

hence,

$$(3.15)\qquad dP^{-1}r^2 = [-2Fr.P^{-1}r + R^{-1}Hr.Hr - \mu Qr^2 - |G^*P^{-1}r|^2]dt$$

$$+ 2P^{-1}r.[F - PH^*R^{-1}H + \mu PQ]rdt$$

$$+ 2P^{-1}r.Bvdt + 2r.H^*R^{-1}dy$$

$$+ tr(PH^*R^{-1}H)dt$$

$$= [-R^{-1}Hr.Hr + \mu Qr^2 - |G^*P^{-1}r|^2$$

$$+ 2P^{-1}r.Bv + tr(PH^*R^{-1}H)]dt$$

$$+ 2r.H^*R^{-1}dy.$$

Moreover writing

$$\dot{P}' = [F + PF^*P^{-1} - PH^*R^{-1}H + GG^*P^{-1} + \mu PQ]P$$

one deduces

$$(3.19) \qquad d\ln|P(t)|/dt = tr(F+PF^*P^{-1}-PH^*R^{-1}H+GG^*P^{-1}+\mu PQ)$$
$$= tr(2F-PH^*R^{-1}H+GG^*P^{-1}+\mu PQ).$$

From (3.12), (3.15), and (3.16), one easily deduces (3.14). □

The assumption (3.8) for $t = t_1$ implies that

$$P^{-1}(t_1) - \mu M > 0$$

is positive definite. Hence one can calculate the integral

$$\int \exp(-\tfrac{1}{2}P^{-1}(t_1)(x-r_{t1})^2 + (\mu/2)Mx^2)dx$$
$$= \exp((\mu/2)r_{t1}.[I - \mu MP(t_1)]^{-1}Mr_{t1})$$
$$(2\pi)^{n/2}|P(t_1)|^{\frac{1}{2}} \, | [I - \mu MP(t_1)]|^{-\frac{1}{2}}.$$

Therefore one can write

$$(3.17) \qquad K(v(\cdot)) = E[\mu \int \exp((\mu/2)Mx^2)\pi(x,t_1)dx]$$
$$= E[\mu \exp[\int_0^{t_1} R^{-1}Hr.dy - \tfrac{1}{2}\int_0^{t_1} H^*R^{-1}Hr.rdt$$
$$+ (\mu/2)\int_0^{t_1} (Qr^2+Nv^2)dt$$
$$+ (\mu/2)[I - \mu MP(t_1)]^{-1}Mr_{t1}.r_{t1}]]$$
$$\exp((\mu/2)\int_0^{t_1} tr(PQ)dt) \, | [I - \mu MP(t_1)]|^{-\frac{1}{2}}$$

assuming that the expectation is finite.

Equation for π^v and its adjoint

To show that K(v(·)) is an alternative expression for the cost, an equation for π^v and its adjoint are needed.

LEMMA 3.4. *The process* $\pi^v(x,t)$ *has the* Ito *differential*

$$
\begin{aligned}
(3.18)\qquad d\pi &= [\tfrac{1}{2}tr(GG^*D_x^2\pi) - (Fx+Bv).D_x\pi \\
&+ (\mu/2)\pi(Qx^2+Nv^2) - \pi tr(F)]dt \\
&+ \pi R^{-1}Hx.dy \\
&= \pi[(P^{-1}Fx.x + (\mu/2)Qx.x + \tfrac{1}{2}|G^*P^{-1}x|^2 \\
&+ x.(P^{-1}Bv - F^*P^{-1}r - P^{-1}GG^*P^{-1}r) \\
&- \tfrac{1}{2}tr(G^*P^{-1}G + 2F) + \tfrac{1}{2}|G^*P^{-1}r|^2 \\
&+ (\mu/2)Nv^2 - P^{-1}Bv.r)dt \\
&+ R^{-1}Hx.dy]
\end{aligned}
$$

PROOF. This follows by simple calculations from (3.13). □

At this stage it is convenient to use the robust form of (3.18). In order to derive it, it is however necessary to assume that

$$(3.19)\qquad R^{-1}(t)H(t)$$

is differentiable. Let us consider $q^v: R^n \times \Omega \times T \to R$

$$q^v(x,t) = \pi^v(x,t)\exp(-y_t.R^{-1}(t)H(t)x).$$

LEMMA 3.5. *The process* $q^v(\cdot,\cdot)$ *satisfies the equation*

$$
\begin{aligned}
(3.20)\qquad \partial q(x,t)/\partial t &= \tfrac{1}{2}tr(GG^*D_x^2q) + D_{xq}.(GG^*H^*R^{-1}y-Fx-Bv) \\
&+ \tfrac{1}{2}q[|G^*H^*R^{-1}y|^2 + \mu(Qx^2+Nv^2) \\
&- H^*R^{-1}Hx^2 - 2(Fx+Bv).H^*R^{-1}y - 2(R^{-1}H)'x.y - 2tr(F)]
\end{aligned}
$$

PROOF. One has

$$dq(x,t) = d\pi \exp(-y_t . R^{-1}Hx) + \pi \exp(-y_t . R^{-1}Hx)[-dy . R^{-1}Hx - y_t . (R^{-1}H)'xdt] - \tfrac{1}{2}\pi \exp(-y_t . R^{-1}Hx)H^*R^{-1}Hx . xdt$$

from which one easily derives (3.20). □

Next one derives the adjoint equation of (3.20) with respect to (3.4), which reads

$$\text{(3.21)} \quad -\partial p(x,t)/\partial t = \tfrac{1}{2}\mathrm{tr}(GG^*D_x^2 p - D_x p . (GG^*H^*R^{-1}y - Fx - Bv) + \tfrac{1}{2}p[|G^*H^*R^{-1}y|^2 + \mu(Qx^2 + Nv^2) - H^*R^{-1}Hx^2 - 2(Fx+Bv) . H^*R^{-1}y - 2(R^{-1}H)'x . y],$$

$$p(x,t_1) = \mu \exp((\mu/2)Mx^2 + y_{t1} . R^{-1}(t_1)H(t_1)x).$$

In fact it is possible to solve (3.21) exactly.

<u>LEMMA 3.6</u>. *Assume that the Riccati equation (3.5) has a symmetry bounded solution. Define*

$\sigma: \Omega \times T \to R^n$

$$\text{(3.22)} \quad \dot{\sigma} + (F^* - \Sigma GG^*)(\sigma - H^*R^{-1}y) - \Sigma Bv - (H^*R^{-1})'y = 0, \quad \sigma(t_1) = H^*(t_1)R^{-1}(t_1)y_{t1};$$

$\rho: \Omega \times T \to R$

$$\text{(3.23)} \quad \dot{\rho} = -\mathrm{tr}(GG^*\Sigma) + |G^*\sigma|^2 - 2\sigma . (GG^*H^*R^{-1}y - Bv) + [|G^*H^*R^{-1}y|^2 + \mu Nv^2 - 2Bv . H^*R^{-1}y], \quad \rho(t_1) = 0.$$

Then

$$p(x,t) = \mu \exp(-\tfrac{1}{2}[\Sigma(t)x . x - 2\sigma(t) . x + \rho(t)])$$

is a solution of (3.21).

PROOF. One has

$$\partial p(x,t)/\partial t = p(x,t)[-\tfrac{1}{2}\dot{\Sigma}x.x + \dot{\sigma}.x - \tfrac{1}{2}\dot{\rho}],$$

$$D_x p = p[-\Sigma x + \sigma],$$

$$D_x^2 p = p[-\Sigma x + \sigma]\otimes[-\Sigma x + \sigma] - p\Sigma.$$

Substitution in (3.21) yields

$$\begin{aligned}\tfrac{1}{2}\dot{\Sigma}x.x - \dot{\sigma}.x + \tfrac{1}{2}\dot{\rho} = &-\tfrac{1}{2}\mathrm{tr}(GG^*\Sigma) + \tfrac{1}{2}|G^*(-\Sigma x+\sigma)|^2 \\ &-(-\Sigma x+\sigma).(GG^*H^*R^{-1}y-Fx-Bv) \\ &+ \tfrac{1}{2}[|G^*H^*R^{-1}y|^2 + \mu(Qx^2+Nv^2) \\ &- H^*R^{-1}Hx^2 - 2(Fx+Bv).H^*R^{-1}y \\ &- 2(R^{-1}H)'x.y]\end{aligned}$$

and the result follows with (3.5), (3.22), and (3.23). □

LEMMA 3.7. *The functional* K(v(·)) *defined by (3.4) can be calculated by*

(3.24) $$K(v(\cdot)) = E[\int p(x,0)\pi(x,0)dx].$$

PROOF. By the expression for $p(x,t_1)$ in (3.21) one has

$$\mu\int \exp((\mu/2)Mx^2)\pi(x,t_1)dx = \int p(x,t_1)q(x,t_1)dx.$$

This integral makes sense by assumption (3.8). An integration by parts yields

$$\mu\int \exp((\mu/2)Mx^2)\pi(x,t_1)dx = \int p(x,0)q(x,0)dx = \int p(x,0)\pi(x,0)dx.$$

Taking the expectation one deduces (3.24). □

Equality of the two costs

3.8. PROOF OF THEOREM 3.2. From (3.11) and (3.21) follows that

$$J(v(\cdot)) = E[p(x_{t_1},t_1)\lambda_{t_1} z_{t_1} \exp(-y_{t_1}.R^{-1}(t_1)H(t_1)x_{t_1})].$$

But

$$y_{t_1}.R^{-1}(t_1)H(t_1)x_{t_1} = \int_0^{t_1} R^{-1}Hx.dy$$

$$+ \int_0^{t_1} y.[(R^{-1}H)'x + R^{-1}H(Fx+Bv)]dt$$

$$+ \int_0^{t_1} y.R^{-1}HGdw,$$

hence

$$J(v(\cdot)) = E[p(x_{t_1},t_1)\lambda_{t_1} \exp(- \int_0^{t_1} y.((R^{-1}H)'x$$

$$+ R^{-1}H(Fx+Bv))dt$$

$$- \tfrac{1}{2} \int_0^{t_1} H^*R^{-1}Hx.xdt - \int_0^{t_1} y.R^{-1}HGdw)].$$

Attention will be concentrated on

$$X = E[p(x_{t_1},t_1)\lambda_{t_1} \exp(-[\int_0^{t_1} y.((R^{-1}H)'x$$

$$+ R^{-1}H(Fx+Bv))dt + \tfrac{1}{2} \int_0^{t_1} H^*R^{-1}Hx.xdt$$

$$+ \int_0^{t_1} y.R^{-1}HGdw]) | Y^{t_1}].$$

Recall that y,w,x_0 are independent objects. Since v is adapted to Y, one can calculate X by freezing the values of y and v, and taking the expectation with respect to the remaining source of noise, namely w.

Note that for y and v frozen, $p(\cdot,\cdot)$ is a $C^{2,1}$ deterministic function. Therefore

$$(3.25) \qquad \begin{aligned} dp(x_t,t) &= [\partial p(x,t)/\partial t + D_x p.(Fx_t+Bv_t) + \tfrac{1}{2}tr(D_x^2 pGG^*)]dt + D_x p.Gw \\ &= [D_x p.GG^*H^*R^{-1}y - \tfrac{1}{2}p(|G^*H^*R^{-1}y|^2 + \\ &+ \mu(Qx_t^2 + Nvt^2) - H^*R^{-1}Hx_t^2 \\ &- 2(Fx_t + Bvt).H^*R^{-1}y \\ &- 2(R^{-1}H)'x_t.y_t)] + D_x p.Gdw. \end{aligned}$$

Next

$$(3.26) \qquad \begin{aligned} d(p(x_t,t)\lambda_t) &= \lambda_t[D_x p.GG^*H^*R^{-1}y - \tfrac{1}{2}p(|G^*H^*R^{-1}y|^2 \\ &- H^*R^{-1}Hx_t^2 - 2(Fx_t + Bv_t).H^*R^{-1}y_t \\ &- 2(R^{-1}H)'x_t.y_t)]dt \\ &+ \lambda_t D_x p.Gdw_t. \end{aligned}$$

Let us denote

$$\begin{aligned} \theta_t = \exp-[&\int_0^t y.((R^{-1}H)'x_t + R^{-1}H(Fx+Bv))dt \\ &+ \tfrac{1}{2}\int_0^t R^{-1}H^*Hx.xdt + \int_0^t y.R^{-1}HGdw_t], \end{aligned}$$

hence

$$(3.27) \qquad \begin{aligned} d\theta_t = \theta_t[&-y.((R^{-1}H)'x + (R^{-1}H)(Fx+Bv))dt \\ &+ \tfrac{1}{2}H^*R^{-1}Hx_t^2dt - y_t.R^{-1}HGdw_t \\ &+ \tfrac{1}{2}|G^*H^*R^{-1}y|^2dt]. \end{aligned}$$

Combining (3.26) and (3.27) one obtains

$$dp(x_t,t)\lambda_t\theta_t = [\lambda_t\theta_t D_x p + \lambda_t\theta_t H^* R^{-1} y_r].Gdw_t.$$

Recalling that y is frozen, one can take expectation with respect to w, hence

$$X = E[p(x_0,0)|Y^{t_1}] = \int p(x,0)\pi(x,0)dx$$

by x_0 independent of y with a Gaussian distribution and by definition of $\pi(x,0)$. Thus

$$J(v(\cdot)) = E[X] = E[\int p(x,0)\pi(x,0)dx] = K(v(\cdot))$$

by (3.24). □

REMARK 3.9. Assumption (3.19) is not necessary to prove theorem 3.2. because of the following argument. One first approximates $R^{-1}H$ by a differentable function. In this case the preceding proof shows that theorem 3.2. holds. Secondly one observes that the final result does not depend on the derivative and that hence one can pass to the limit. Condition (3.19) is thus only an intermediary technical assumption.

4. SOLUTION OF THE STOCHASTIC CONTROL PROBLEM

The result of theorem 3.2 implies that minimization of the cost functional $J(v(\cdot))$ is equivalent to the minimization of the cost functional $K(v(\cdot))$. The importance of this lies in the fact that the minimization of $K(v(\cdot))$ appears as a stochastic control problem with full information specified by the state equation (3.2) and cost functional (3.4). This problem is now easily solved.

It should however be pointed out that the function π defined in (3.3) is in general not the conditional density of x given y because it depends on the parameters Q and N of the cost functional. A key point in the proof of theorem 3.2 is that one can push the cost functional into the expression for the conditional density. The fact that this is possible is based on the exponential form of the Gaussian density and the cost functional.

Consider the Riccati equation for $S: T \to R^{n\times n}$

(4.1)
$$\begin{cases} \dot{S} + S(F+\mu PQ) + (F^*+\mu QP)S + Q - S(BN^{-1}B^*-\mu PH^*R^{-1}HP)S = 0, \\ S(t_1) = \tfrac{1}{2}[(I-\mu MP(t_1))^{-1}M + M(I-\mu P(t_1)M)^{-1}]. \end{cases}$$

It will be assumed that (4.1) has a symmetric solution. Consider the equation

(4.2)
$$d\hat{r} = [F - PH^*R^{-1}H + \mu PQ - BN^{-1}B^*S]\hat{r}\,dt + PH^*R^{-1}dy, \quad \hat{r}_0 = \mu_0,$$

which corresponds to the state equation (3.2) with the control

(4.3)
$$u_t^* = -N^{-1}(t)B^*(t)S(t)\hat{r}_t.$$

For some r given by (3.2) define

(4.4)
$$\begin{aligned} h_t = & \int_0^t R^{-1}Hr.dy - \tfrac{1}{2}\int_0^t H^*R^{-1}Hr.rds \\ & + (\mu/2)\int_0^t (Qr^2+Nv^2)ds + (\mu/2)S(t)r_t.r_t \\ & - (\mu/2)\int_0^t tr(SPH^*R^{-1}HP)ds. \end{aligned}$$

Attention is now restricted from the dass of admissable controls $\underline{U}_2$ to

$$\underline{U}_3 = \{u \in \underline{U}_2 \mid E[\int_0^{t_1} \exp(2h_s)|R^{-1}H(I+\mu PS)r|^2 ds] < \infty\}.$$

THEOREM 4.1. *Assume the conditions of theorem 3.2, and that a symmetric solution to (4.1) exists. Assume that* u^**, defined by (4.3), belongs to the class of admissable controls, and moreover to* $\underline{U}_3$.

a. *Then* u^* *is optimal and*

$$\begin{aligned} \min_{v\in\underline{U}_3} J(v(\cdot)) &= J(u^*(\cdot)) \\ &= \mu \exp((\mu/2)[S_0\mu_0.\mu_0 + \int_0^{t_1} tr(PQ+SPH^*R^{-1}HP)\,ds])\,|[I - \mu MP(t_1)]|^{-\frac{1}{2}}. \end{aligned}$$

b. *Then also* u^* *is conditionally optimal in* $\underline{U}_3$.

REMARKS 4.2. 1. In the representation of the solution as given by (3.1), (4.2), and (4.3), the separation property does not hold in general. Note that in the sufficient statistic π for the cost functional both r and P depend on the state cost matrix Q, while the control Riccati differential equation for S depends on the matrix function P.

2. Observe that the function $\pi^v(x,t)$ is in general not the conditional density of x_t given Y^t since its parameters r and P depend on the cost functional through Q.
3. The concept of a sufficient statistic for a stochastic control problem has been defined by C. Striebel [8, 3.2]. For the stochastic control problem under consideration it follows from the proof of theorem 4.1 that $\hat{r}$, as defined by (4.2), is a sufficient statistic. Note that because P is a deterministic function it is therefore considered not to be a sufficient statistic.
4. An attempt to define minimality of a sufficient statistic for a stochastic control problem will not be made here. Because $\hat{r}$ takes values in R^n, the state space of the given stochastic system, it seems likely that this sufficient statistic is minimal in any reasonable sense. In the special case of G = 0 a sufficient statistic of much higher dimension has been found in Speyer et al. [6] for discrete-time systems, and in Kumar, van Schuppen [5] for continuous-time systems.
5. Theorem 4.1 contains the special cases discussed by Speyer et al. [6], with Q = 0, and by Kumar, van Schuppen [5], with G = 0. The discrete-time case of the problem considered here is discussed by Whittle [10].
6. When μ is small one has that

$$[J_\mu(v(\cdot)) - \mu]/\mu^2 \to \tfrac{1}{2}\tilde{E}^v[\int_0^{t_1}(Qx^2+Nv^2)ds + Mx_{t_1}^2].$$

Hence for μ small the LEG stochastic control problem becomes close to the standard linear-quadratic-Gaussian stochastic control problem for which the separation principle holds. This can also be seen from the explicit expressions for the optimal control. For μ small, S becomes close to the solution of the Riccati differential equation of the deterministic linear quadratic control problem

$$\dot{\Pi} + \Pi F + F^{*}\Pi + Q - \Pi B N^{-1} B^{*}\Pi = 0, \quad \Pi(t_1) = M.$$

Moreover, then (3.1) reduces to the Riccati equation of the Kalman filter, and (3.2) reduces to the Kalman filter itself.

A sufficient condition for conditional optimality that will be used in the proof of theorem 4.1, will be stated.

THEOREM 4.3. [C. Striebel] *If there exists a* $u^{*} \in \underline{U}_3$, *and for any admissable control* $v \in \underline{U}_3$ *an* Y^t *adapted process* $h^v: \Omega \times T \to R$ *such that*

1. *for any* $v \in \underline{U}_2$ $\tilde{E}^v[c|Y^{t_1}] = h^v_{t_1}$, *and* h^v *is a submartingale on* Y^t *with respect to* $\tilde{P}^v$;
2. h^{v^*} *is a* Y^t *martingale with respect to* $\tilde{P}^{v^*}$;

then v^* *is conditionally optimal. One calls* h^v *the conditional cost functional associated with* v.

PROOF OF 4.1. a. By theorem 3.2 $J(v(\cdot)) = K(v(\cdot))$. Recall that

$$h_t = \int_0^t R^{-1}Hr.dy - \tfrac{1}{2}\int_0^t H^{*}R^{-1}Hr.rds + (\mu/2)\int_0^t (Qr^2+Nv^2)ds + (\mu/2)S(t)r_t.r_t - (\mu/2)\int_0^t tr(SPH^{*}R^{-1}HP)ds.$$

Then

$$K(v(\cdot)) = \mu \exp((\mu/2)\int_0^{t_1} tr(PQ+SPH^{*}R^{-1}HP)ds) \tag{4.5}$$

$$|[I - \mu MP(t_1)]|^{-\frac{1}{2}} E[\exp(h_{t_1})].$$

Calculations show that

$$dh_t = R^{-1}Hr.dy - \tfrac{1}{2}H^{*}R^{-1}Hr.rdt + (\mu/2)(Qr^2+Nv^2)dt + (\mu/2)\dot{S}r.rdt + \mu Sr.[(F-PH^{*}R^{-1}H+\mu PQ)r + Bv]dt + \mu Sr.PH^{*}R^{-1}dy,$$

$$(4.6)\qquad d\exp(h_t) = \exp(h_t)[R^{-1}H(I+\mu PS)r.dy - \tfrac{1}{2}H^*R^{-1}Hr.rdt + (\mu/2)(Qr^2+Nv^2)dt + (\mu/2)\dot{S}r.r\,dt + \mu Sr.[(F-PH^*R^{-1}H + \mu PQ)r + Bv]dt + \tfrac{1}{2}|R^{-\frac{1}{2}}H(I+\mu PS)r|^2 dt]$$

$$= \exp(h_t)[R^{-1}H(I+\mu PS)r.dy + (\mu/2)(\dot{S}+S(F+\mu PQ) + (F^*+\mu QP)S - S(BN^{-1}B^*-\mu PH^*R^{-1}HP)S + Q)r.rdt + (\mu/2)N(v+N^{-1}B^*Sr)^2 dt].$$

If S satisfies (4.1) then

$$(4.7)\qquad \exp(ht_1) \geq \exp((\mu/2)S_0r_0.r_0) + \int_0^{t_1} \exp(h_s)R^{-1}H(I+\mu PS)r.dy$$

and from (4.5) and the definition of $\underline{U}_3$ then follows that

$$(4.8)\qquad K(v(\cdot)) \geq \mu \exp((\mu/2)[S_0\mu_0.\mu_0 + \int_0^{t_1} tr(PQ+SPH^*R^{-1}HP)ds]) \,|\,[I-\mu MP(t_1)]|^{-\frac{1}{2}}.$$

For the control u^* defined by (4.3) one gets equality in (4.7) and (4.8), hence it is optimal and

$$\min_{u\in\underline{U}_2} J(v(\cdot)) = \min_{u\in\underline{U}_2} K(v(\cdot)) =$$

$$= \mu \exp((\mu/2)[S_0\mu_0.\mu_0 + \int_0^{t_1} tr(PQ+SPH^*R^{-1}HP)ds])$$

$$|[I - \mu MP(t_1)]|^{-\frac{1}{2}}.$$

b. Let

$$c = \mu \exp((\mu/2)[Mx_{t_1}^2 + \int_0^{t_1} (Qx^2+Nv^2)ds]),$$

$$\rho_t = \exp(\int_0^t R^{-1}Hx.dy - \tfrac{1}{2}\int_0^t H^*R^{-1}Hx.xds),$$

$$\bar{\rho}_t = \exp(\int_0^t R^{-1}Hr.dy - \tfrac{1}{2}\int_0^t H^*R^{-1}Hr.rds),$$

$$h_t = \mu\ a_t \exp((\mu/2)[S_t r_t.r_t + \int_0^t (Qr^2+Nv^2)ds]),$$

$$a_t = \exp((\mu/2)[\int_0^{t_1} tr(PQ)ds + \int_t^{t_1} (PH^*R^{-1}HPS)ds])$$
$$|[I-\mu MP(t_1)]|^{-\frac{1}{2}},$$

$$k_t = h_t\ \bar{\rho}_t.$$

By the proof of theorem 3.2

$$E[c\rho_{t_1}|Y^{t_1}] = X = \int p(x,0)\pi(x,0)dx,$$

which by the proof of lemma 3.7 equals

$$= \int \mu \exp((\mu/2)Mx^2)\pi(x,t_1)dx.$$

With the calculations above (3.17) one obtains

$$E[c\rho_{t_1}|Y^{t_1}] = k_{t_1} = h_{t_1}\bar{\rho}_{t_1}.$$

Setting in this expression M = 0, Q = 0, N = 0, it materializes that

$$E[\mu\rho_{t_1}|F^{y_1}] = \mu\bar{\rho}_{t_1},$$
$$\tilde{E}^v[c|Y^{t_1}] = E[c\rho_{t_1}|Y^{t_1}]/E[\rho_{t_1}|Y^{t_1}] = h_{t_1}.$$

It is claimed that if k is a submartingale with respect to P that then h is a submartingale with respect to $\tilde{P}^v$. For if $s,t \in T$, $s < t$, then

$$\tilde{E}^v[h_t|Y^s] = E[h_t\rho_{t_1}|Y^s]/E[\rho_{t_1}|Y^s]$$

$$= E[h_tE[\bar{\rho}_{t_1}|Y^{t_1}]|Y^s]/E[E[\bar{\rho}_{t_1}|Y^{t_1}]|Y^s]$$

$$= E[h_tE[\bar{\rho}_{t_1}|Y^t]|Y^s]/E[\bar{\rho}_{t_1}|Y^s]$$

$$= E[h_t\bar{\rho}_t|Y^s]/\bar{\rho}_s \geq h_s\bar{\rho}_s/\bar{\rho}_s = h_s.$$

It is then clear that if k is martingale with respect to P that then also h is a martingale with respect to $\tilde{P}^v$.

Using the fact that S satisfies (4.1) and that u^* is given by (4.3), a lengthy calculation shows that

$$dk = d(h\bar{\rho}) = (\mu^2/2)a_t \exp((\mu/2)$$
$$[S_tr_t.r_t + \int_0^t (Qr^2+Nv^2)ds])$$
$$[N(v_t-u_t^*)^2dt + 2(R^{-1}Hr/\mu+R^{-1}HPSr).dy].$$

Thus for any $v \in \underline{U}_3$, $k = h\bar{\rho}$ is a submartingale with respect to P, and for $v = u^*$ a martingale. By the above claim h is then for any $v \in \underline{U}_2$ a $\tilde{P}^v$ submartingale and for $v = u^*$ a martingale. From 4.3 then follows that u^* is conditionally optimal. □

Note that in the proof of theorem 4.1 a key element is the invariance of the conditional cost functional h.

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